Computational and Communication Complexity in Massively Parallel Computation

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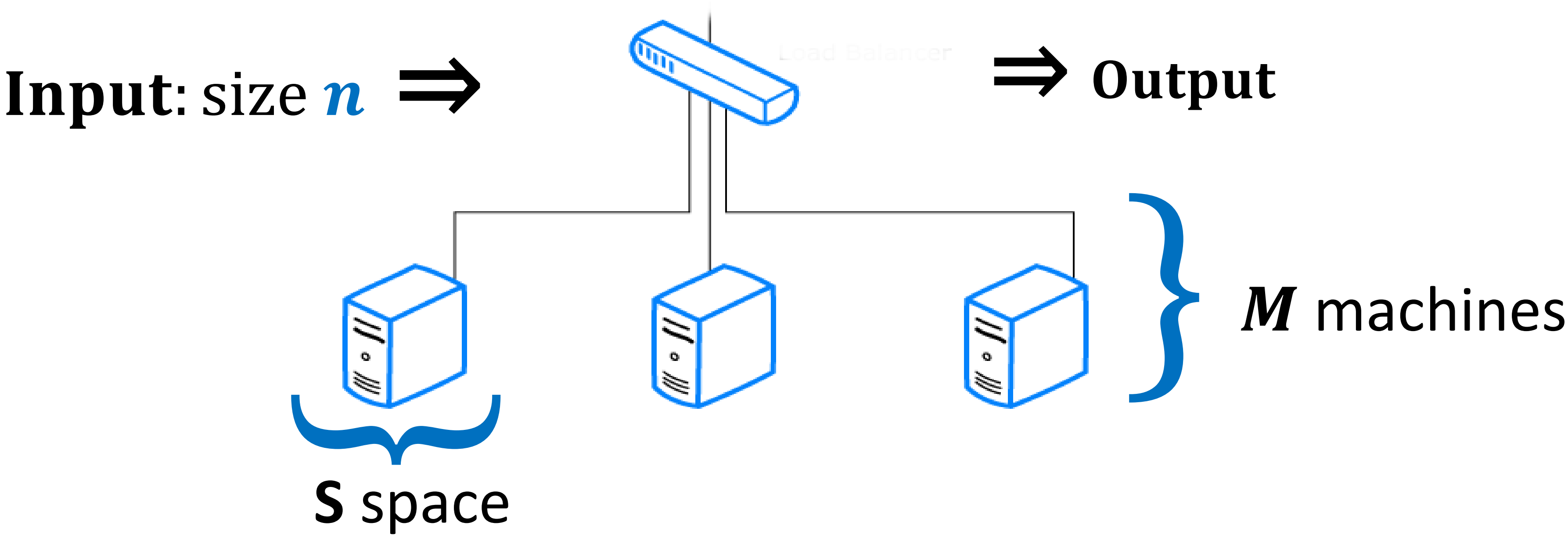
Cluster Computation (a la BSP)

- **Input**: size **n** (e.g. **n** = billions of edges in a graph)
- $\boldsymbol{M}$ **M**achines, $\boldsymbol{S}$ **S**pace (RAM) each
  - Constant overhead in RAM: $\boldsymbol{M} \cdot \boldsymbol{S} = O(\boldsymbol{n})$
  - $\boldsymbol{S}$ = $\boldsymbol{n}^{1-\epsilon}$, e.g. ϵ = 0.1 or ϵ = 0.5 ($\boldsymbol{M} = \boldsymbol{S} = O(\sqrt{\boldsymbol{n}})$)
- **Output**: solution to a problem (often size O($\boldsymbol{n}$))
  - Doesn't fit in local RAM ($\boldsymbol{S} \ll \boldsymbol{n}$)

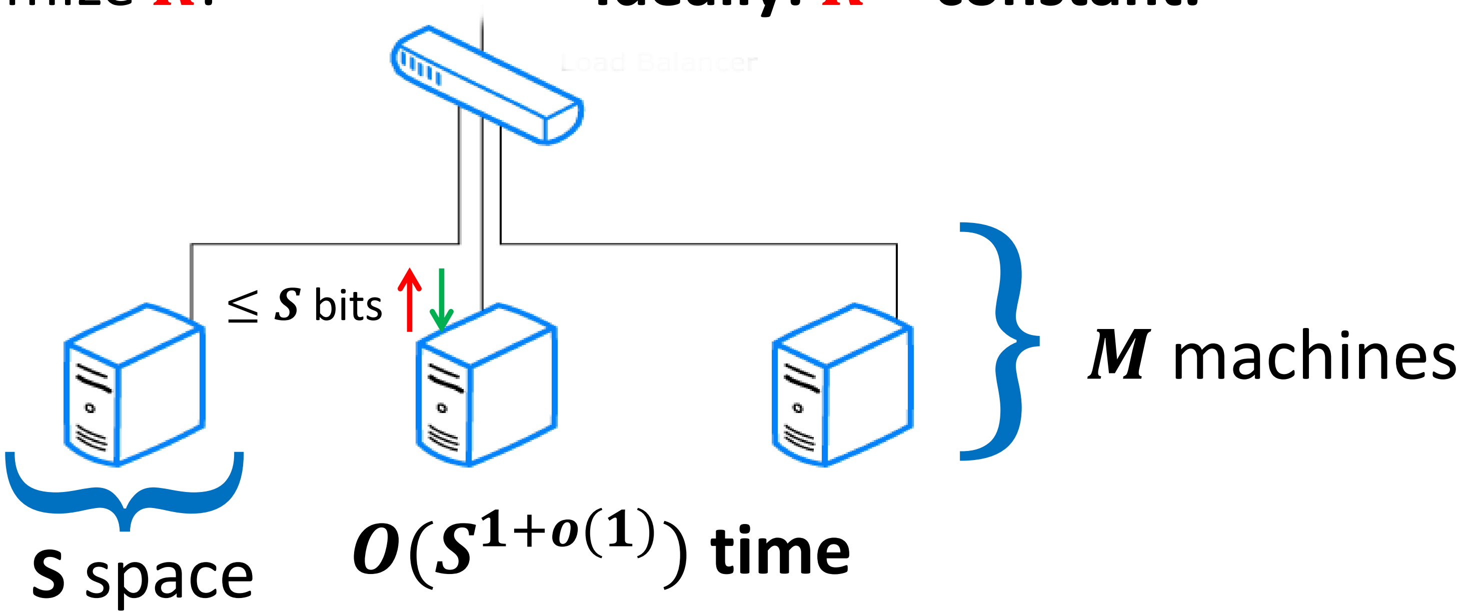


Cluster Computation (a la BSP)

- Computation/Communication in $\boldsymbol{R}$ rounds:
  - Every machine performs a **near-linear time** computation => Total user time $O(\boldsymbol{S}^{\mathbf{1+o(1)}}\boldsymbol{R})$
  - Every machine **receives at most $\boldsymbol{S}$ bits** of information => Total communication $O(\boldsymbol{nR})$.

Goal: Minimize $\boldsymbol{R}$.

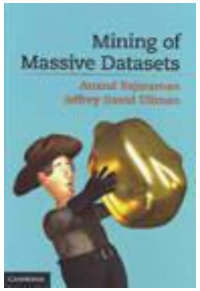
Ideally: $\boldsymbol{R}$ = constant.



MapReduce-style computations

What I won't discuss today

- PRAMs (**shared memory**, multiple processors) (see e.g. [Karloff, Suri, Vassilvitskii'10])
  - Computing XOR requires $\tilde{\Omega}(\log n)$ rounds in CRCW PRAM
  - Can be done in $O(\log_s n)$ rounds of MapReduce
- Pregel-style systems, Distributed Hash Tables (see e.g. Ashish Goel's class notes and papers)
- Lower-level implementation details (see e.g. Rajaraman-Leskovec-Ullman book)



Models of parallel computation

- **Bulk-Synchronous Parallel Model** (BSP) **[Valiant,90]**

  Pro: Most general, generalizes all other models

  Con: Many parameters, hard to design algorithms

- **Massive Parallel Computation** [Feldman-Muthukrishnan-Sidiropoulos-Stein-Svitkina'07, Karloff-Suri-Vassilvitskii'10, Goodrich-Sitchinava-Zhang'11, Beame, Koutris, Suciu'13**, Andoni, Onak, Nikolov, Y.'14**]

  Pros:

  - Inspired by **modern** systems (MapReduce, Dryad, Spark, Giraph, …)
  - Few parameters, **simple** to design algorithms
  - **New algorithmic ideas**, robust to the exact model specification
  - **# Rounds** is an information-theoretic measure => can prove unconditional results

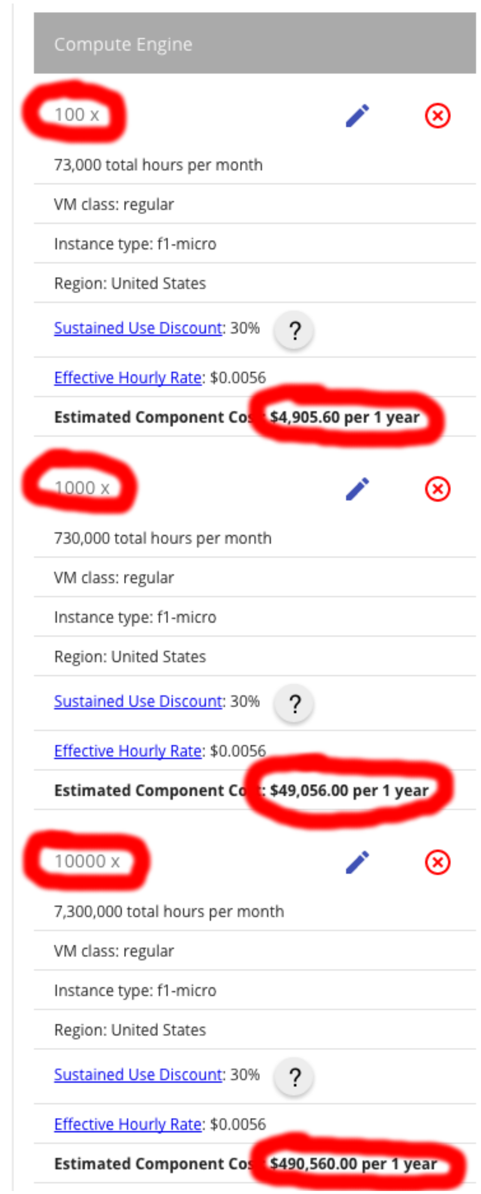
  Con: sometimes not enough to model more complex behavior

Getting hands dirty

- Cloud computing platforms (all offer free trials):
  - Amazon EC2 (1 CPU/12mo)
  - Microsoft Azure ($200/1mo)
  - Google Compute Engine ($200/2mo)
- Distributed Google Code Jam
  - First time in 2015: https://code.google.com/codejam/distributed_index.html
  - Caveats:
    - Very basic aspects of distributed algorithms (few rounds)
    - Small data ($\sim 1\ GB$, with hundreds MB RAM)
    - Fast query access ($\sim 0.01\ ms$ per request), "data with queries"

Business perspective

- Pricings:
  - https://cloud.google.com/pricing/
  - https://aws.amazon.com/pricing/
- ~Linear with **space** and **time** usage
  - 100 machines: 5K $/year
  - 10000 machines: 0.5M $/year
- You pay **a lot more** for using provided algorithms
  - https://aws.amazon.com/machine-learning/pricing/



Sorting: Terasort

- Sort Benchmark: http://sortbenchmark.org/
- Sorting $\boldsymbol{n}$ keys on $\boldsymbol{M} = \boldsymbol{O}(\boldsymbol{n}^{\epsilon})$ machines
  - Would like to partition keys uniformly into blocks: first $\boldsymbol{n}/\boldsymbol{M}$, second $\boldsymbol{n}/\boldsymbol{M}$, etc.
  - Sort the keys locally on each machine
- Build an approximate histogram:
  - Each machine takes a sample of size $\boldsymbol{s}$
  - All $\boldsymbol{M} * \boldsymbol{s} \leq \boldsymbol{S} = \boldsymbol{n}^{\mathbf{1}-\epsilon}$ samples are sorted locally
  - Blocks are computed based on the samples
- By Chernoff: $\mathbf{M} * \boldsymbol{s} = O\left(\frac{\log \boldsymbol{n}}{\boldsymbol{\alpha}^{2}}\right)$ samples suffice to compute all block sizes up to $\pm\boldsymbol{\alpha n}$ error with high probability
- Take $\alpha = \boldsymbol{n}^{-\epsilon}$: error $O(\boldsymbol{S})$
- $\mathbf{M} * \boldsymbol{s} = \widetilde{O}(\boldsymbol{n}^{\mathbf{3}\epsilon}) \leq \boldsymbol{O}(\boldsymbol{n}^{\mathbf{1}-\epsilon})$ for $\epsilon \leq 1/4$

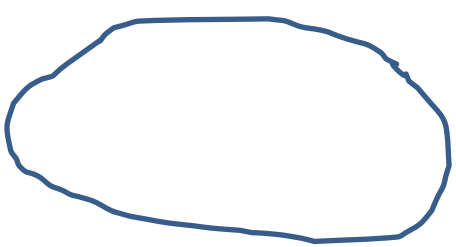
Connectivity

- **Input**: **n** edges of a graph (arbitrarily partitioned between machines)
- **Output**: is the graph connected? (or # of connected components)
- **Question:** how many rounds does it take?
  1. $O(1)$
  2. $O(\log^{\alpha} n)$ ✓
  3. $O(n^{\alpha})$
  4. $O(2^{\alpha n})$
  5. Impossible

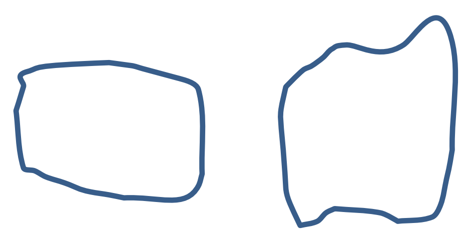
Algorithms for Graphs

- **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg |V|$
    - Linear sketching: one round
    - "Filtering" (Output fits on a single machine) [Karloff, Suri Vassilvitskii, SODA'10; Ene, Im, Moseley, KDD'11; Lattanzi, Moseley, Suri, Vassilvitskii, SPAA'11; Suri, Vassilvitskii, WWW'11]
  - Sparse: $\boldsymbol{S} \ll |V|$ (or $\boldsymbol{S} \ll$ solution size)

    Sparse graph problems appear hard (**Big open question**: connectivity in $o(\log n)$ rounds?)

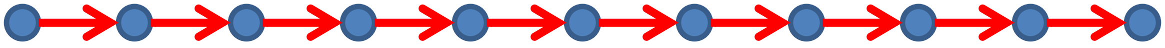


vs.

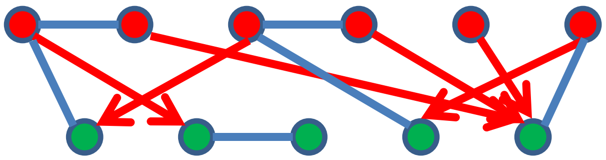


Algorithm for Connectivity

- Version of **Boruvka's algorithm**:
  - All vertices assigned to different components
  - Repeat $O(\log \mathbf{n})$ times:
    - Each component chooses a neighboring component
    - All pairs of chosen components get merged
- How to avoid **chaining**?



- If the graph of components is bipartite and only one side gets to choose then no chaining



- **Randomly** assign components to the sides

Algorithm for Connectivity: Setup

Data: **n** edges of an undirected graph.

Notation:

- $\pi(v) \equiv$ unique id of v
- $\Gamma(S) \equiv$ set of neighbors of a subset of vertices S.

Labels:

- Algorithm assigns label $\ell(v)$ to each v.
- $L_v \equiv$ set of vertices with label $\ell(v)$ (invariant: subset of the connected component containing v).

Active vertices:

- Some vertices will be called **active** (exactly one per L_v).

Algorithm for Connectivity

- Mark every vertex as **active** and let $\ell(v) = \pi(v)$.
- For phases $i = 1,2, \ldots, O(\log \mathbf{n})$ do:
  - Call each **active** vertex a **leader** with probability 1/2. If v is a **leader**, mark all vertices in L_v as **leaders**.
  - For every **active** **non-leader** vertex w, find the smallest **leader** (by π) vertex $w^\star$ in $\Gamma(L_w)$.
  - Mark w **passive**, relabel each vertex with label w by $w^\star$.
- **Output**: set of connected components based on ℓ.

Algorithm for Connectivity: Analysis

- If $\ell(u) = \ell(v)$ then u and v are in the same CC.
- **Claim:** Whp unique labels in CC in $O(\log N)$ phases
- # active vertices reduces by a constant factor:
  - Half of the active vertices declared as non-leaders.
  - Fix an active **non-leader** vertex $\boldsymbol{v}$.
  - If at least two different labels in the CC of v then there is an edge $(\boldsymbol{v'}, \boldsymbol{u})$ such that $\ell(\boldsymbol{v}) = \ell(\boldsymbol{v'})$ and $\ell(\boldsymbol{v'}) \neq \ell(\boldsymbol{u})$.
  - $\boldsymbol{u}$ marked as a **leader** with probability 1/2 $\Rightarrow$ half of the active non-leader vertices will change their label.
  - Overall, expect 1/4 of labels to disappear.

Algorithm for Connectivity: Implementation Details

- Distributed data structure of size $O(|V|)$ to maintain labels, ids, leader/non-leader status, etc.
  - O(1) rounds per stage to update the data structure
- Edges stored locally with all auxiliary info
  - Between stages: use distributed data structure to update local info on edges
- For every **active** **non-leader** vertex w, find the smallest **leader** (w.r.t π) vertex $\mathrm{w}^\star \in \Gamma(L_w)$
  - Each (**non-leader,** **leader**) edge sends an update to the distributed data structure
- Much faster with Distributed Hash Table Service (DHT) [Kiveris, Lattanzi, Mirrokni, Rastogi, Vassilvitskii'14]

MPC and Computation Complexity

- Class MRC^i = solvable in $O(\log^i n)$ rounds of MPC
- MRC = $\cup_i MRC^i$ where union is over all constant i
- [Karloff, Suri, Vassilvitskii SODA'10]
  - If $P \subsetneq NC$ then deterministic $MRC \subsetneq NC$
  - Can simulate t-time CRCW PRAM algorithm in $O(t)$ rounds
- [Jacob, Lieber, Sitchinnava, MFCS'14]
  - Only known unconditional LB: $\Omega(\log n)$ for Guided Interval Fusion
- [Fish, Kun, Lelkes, Reyzin, Turan DISC'15]
  - Can recognize regular languages in O(1) rounds
  - Some (conditional) hierarchy theorems for MPC
- [Roughgarden, Vassilvitskii, Wang SPAA'16]
  - Show $\Omega(\log_S n)$ lower bounds using degree bound
  - Certain type of $\Omega(1)$-round MPC lower bounds implies $P \subsetneq NC^1$

MPC for Specific Problems

- Combinatorial Optimization
  - Matchings
    - Large constant approx. in $O(\log \log^2 n)$ rounds ["6 Poles"]
    - Small constant approximation in $O(\log n)$ rounds
  - Submodular Maximization [BENW, STOC'16]
  - $(1+\epsilon)$-approx. Euclidean Bichromatic Matching Size in O(1) rounds for constant dimension [ANO**Y**'14, STOC'14]
  - $(1+\epsilon)$-approx. Euclidean MST in O(1) rounds for constant dimension [ANO**Y**'14, STOC'14]

MPC for Specific Problems

- Clustering
  - K-means: [BMVKV, VLDB’12][BEL, NIPS’13]
  - K-center, K-median: [EIM, KDD’11]
  - Correlation Clustering: [CDK, KDD’14]
  - Single-Linkage Clustering: [Vadapalli, **Y** ‘17]
- See my talk at Facebook for details on clustering

MPC for Specific Problems

- Dynamic Programming
  - [Im, Moseley, Sun STOC'17]:
    - Optimal Binary Search Tree
    - Weighted Interval Selection
    - Longest Increasing Subsequence
  - Active area of research right now
- Other problems
  - Triangle Counting
  - …

Thanks! Questions?

- Slides will be available on **http://grigory.us**
- More about algorithms for massive data:

  http://grigory.us/blog/
- More in the classes I teach:

CIS 700:
KEEP CALM
AND
CRUNCH
DATA IN o(N)

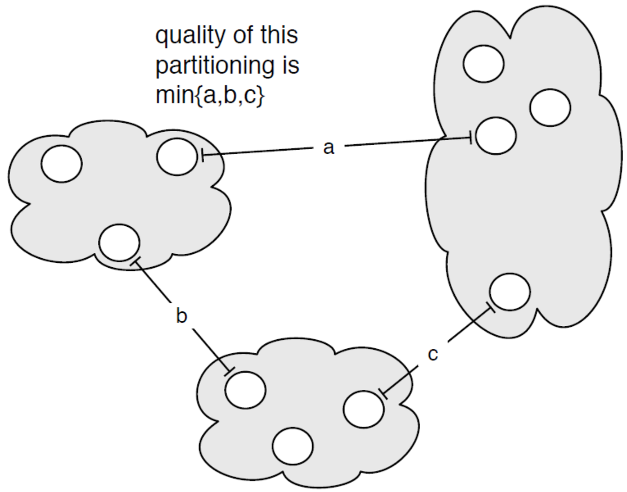
CSCI B609:
KEEP CALM
AND
DIG
FOUNDATIONS
OF
DATA SCIENCE

Example: Single Linkage Clustering

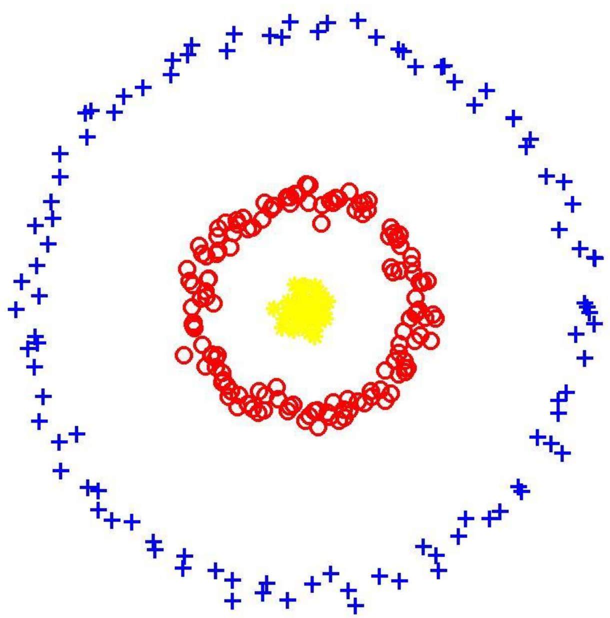
- [Zahn'71] **Clustering** via Minimum Spanning Tree:

k clusters: remove $k-1$ longest edges from MST

- Maximizes **minimum** intercluster distance



[Kleinberg, Tardos]



Large geometric graphs

- Graph algorithms: **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg |V|$.
  - Sparse: $\boldsymbol{S} \ll |V|$.

- **Our setting:**
  - Dense graphs, sparsely represented: O(**n**) space
  - Output doesn't fit on one machine ($\boldsymbol{S} \ll \boldsymbol{n}$)
- **Today:** $(1+\epsilon)$-approximate MST [Andoni, Onak, Nikolov, **Y.**]
  - $\boldsymbol{d} = 2$ (easy to generalize)
  - $\boldsymbol{R} = \log_{\boldsymbol{S}} \boldsymbol{n} =$ O(1) rounds ($\boldsymbol{S} = \boldsymbol{n}^{\boldsymbol{\Omega(1)}}$)

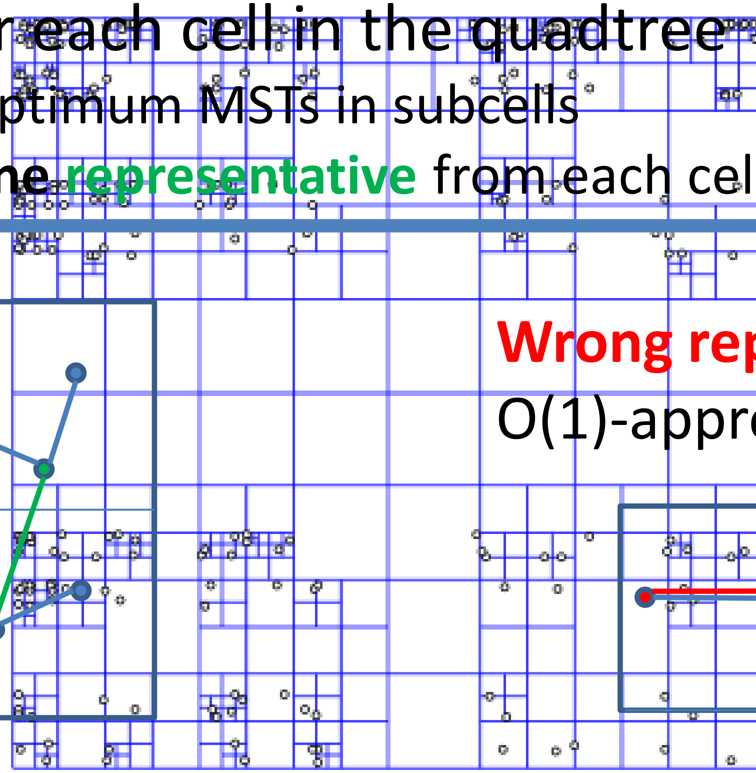
$O(\log n)$-MST in $R = O(\log n)$ rounds

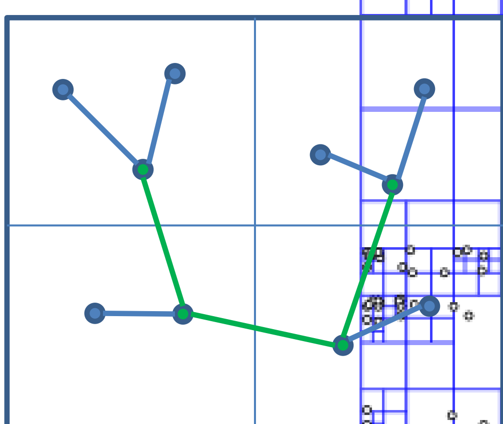
- Assume points have integer coordinates $[0, \ldots, \Delta]$, where $\Delta = O(\boldsymbol{n^2})$.

Impose an $O(\log \boldsymbol{n})$-depth quadtree

Bottom-up: For each cell in the quadtree

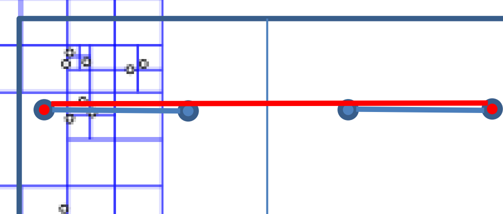
- compute optimum MSTs in subcells
- Use only **one** representative from each cell on the next level





Wrong representative:

O(1)-approximation per level



ϵL-nets

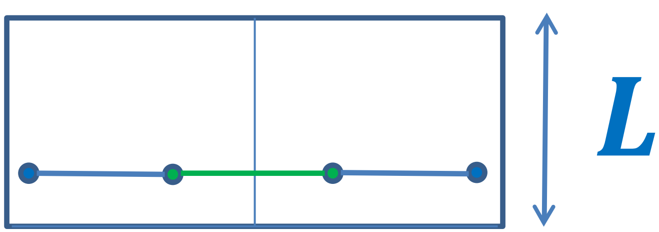
- ϵL-net for a cell C with side length L:

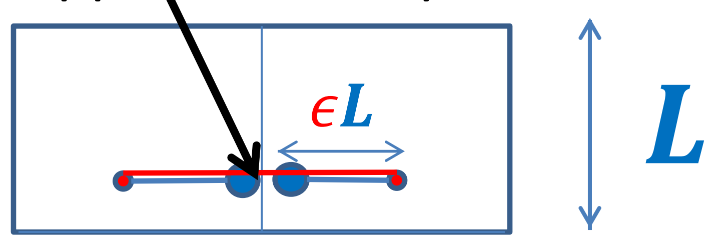
  Collection **S** of vertices in C, every vertex is at distance <= ϵL from some vertex in **S**. (Fact: Can efficiently compute ϵ-net of size $O\left(\frac{1}{\epsilon^2}\right)$)

Bottom-up: For each cell in the quadtree
– Compute optimum MSTs in subcells
– Use ϵL-net from each cell on the next level

- **Idea**: Pay only O(ϵL) for an **edge** cut by cell with side L
- Randomly shift the quadtree:
  $\Pr[cut\ edge\ of\ length\ \ell\ by\ L] \sim \ell/L$ – charge errors

Wrong representative:
O(1)-approximation per level





Randomly shifted quadtree

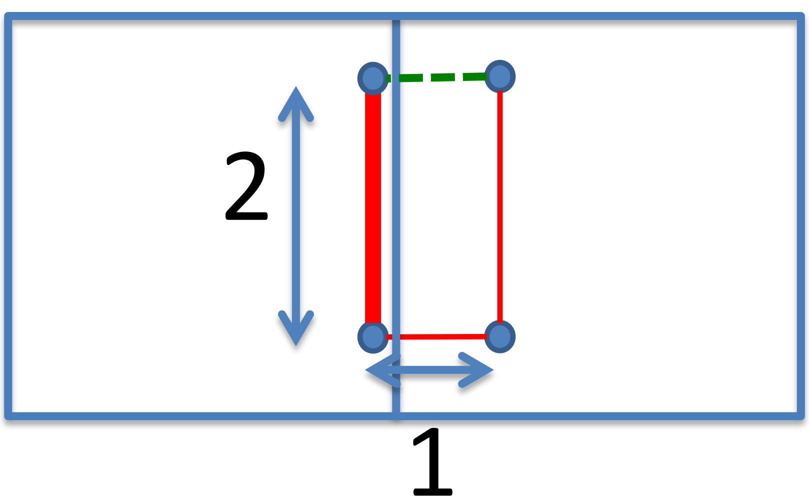
- Top cell shifted by a random vector in $[0, \boldsymbol{L}]^2$

Impose a **randomly shifted** quadtree (top cell length $\mathbf{2\Delta}$)

Bottom-up: For each cell in the quadtree

– Compute optimum MSTs in subcells

– Use $\boldsymbol{\epsilon L}$-net from each cell on the next level



Pay **5** instead of **4**

Bad Cut

Pr[Bad Cut] = Ω(1)

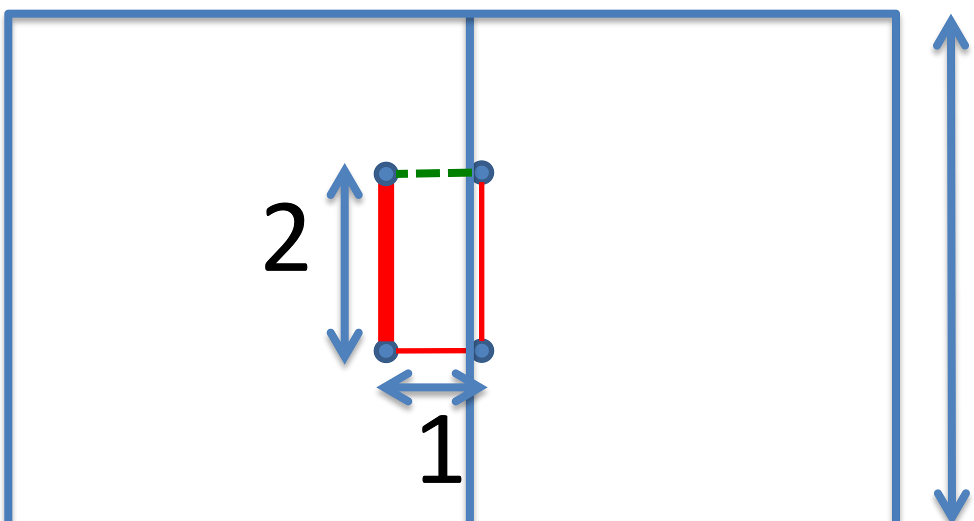
$(1+\epsilon)$-MST in $\mathbf{R} = O(\log\ n)$ rounds

- **Idea:** Only use short edges inside the cells

Impose a **randomly shifted** quadtree (top cell length $\frac{2\Delta}{\epsilon}$)

Bottom-up: For each node (cell) in the quadtree

- compute optimum Minimum Spanning **Forests** in subcells, **using edges of length** $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

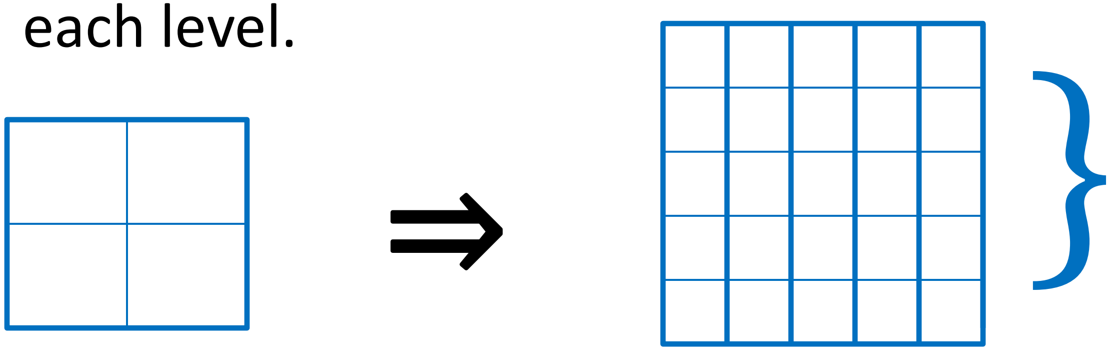


$$L = \Omega(\frac{1}{\epsilon})$$

Pr[Bad Cut] $= O(\epsilon)$

$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

- $O(\log n)$ rounds => $O(\log_s n)$ = O(1) rounds
  - Flatten the tree: $(\sqrt{M} \times \sqrt{M})$-grids instead of (2x2) grids at each level.



$\sqrt{M} = n^{\Omega(1)}$

Impose a **randomly shifted** $(\sqrt{M} \times \sqrt{M})$-tree

Bottom-up: For each node (cell) in the tree

- compute optimum MSTs in subcells via edges of length $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

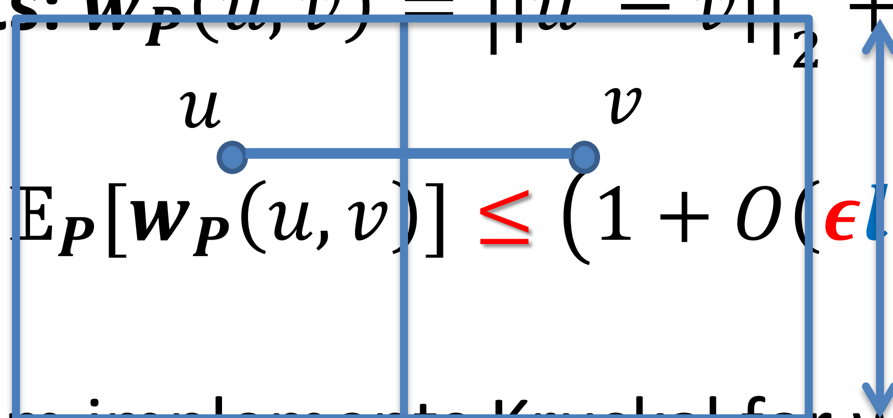
$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

Theorem: Let l = # levels in a random tree P

$$\mathbb{E}_P[\mathbf{ALG}] \le \left(1 + O(\epsilon l d)\right)\mathbf{OPT}$$

Proof (sketch):

- $\Delta_P(u, v)$ = cell length, which first partitions (u, v)
- **New weights:** $w_P(u, v) = \|u - v\|_2 + \epsilon \Delta_P(u, v)$



$$\|u - v\|_2 \le \mathbb{E}_P[w_P(u, v)] \le \left(1 + O(\epsilon l d)\right)\|u - v\|_2$$

- Our algorithm implements Kruskal for weights w_P

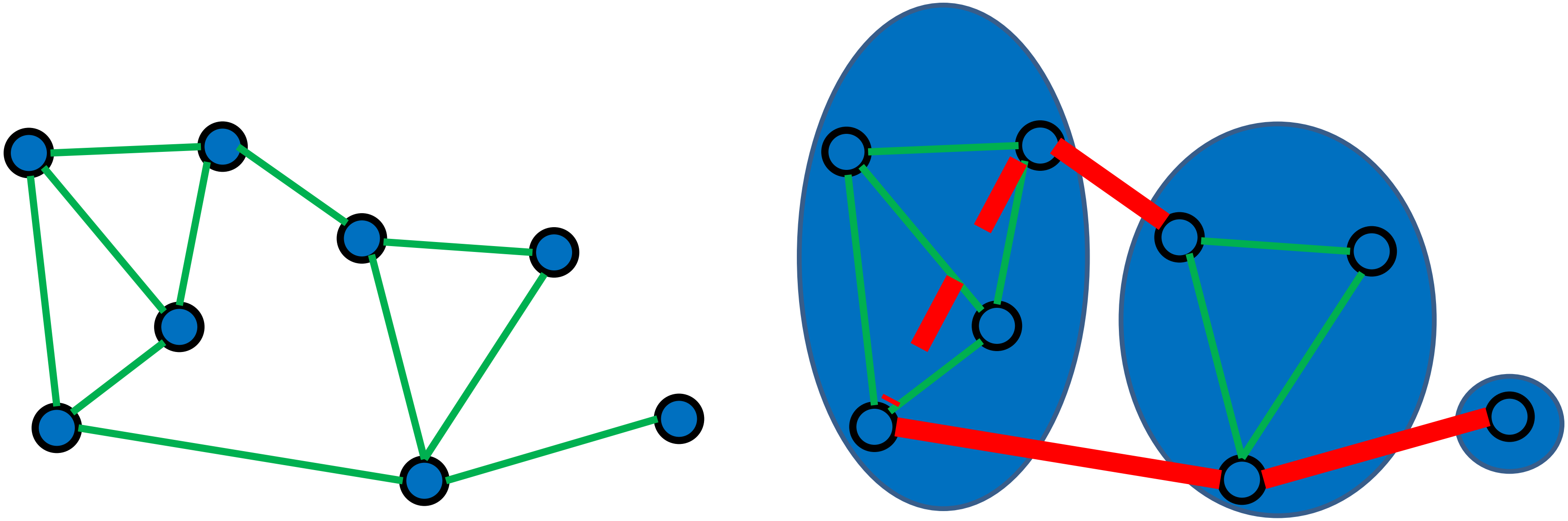
Technical Details

$(1+\epsilon)$-**MST**:

- "**Load balancing**": partition the tree into parts of the same size
- **Almost linear time locally**: Approximate Nearest Neighbor data structure [Indyk'99]
- Dependence on dimension **d** (size of ϵ-net is $O\left(\frac{d}{\epsilon}\right)^{d}$)
- Generalizes to bounded **doubling dimension**

Problem 2: Correlation Clustering

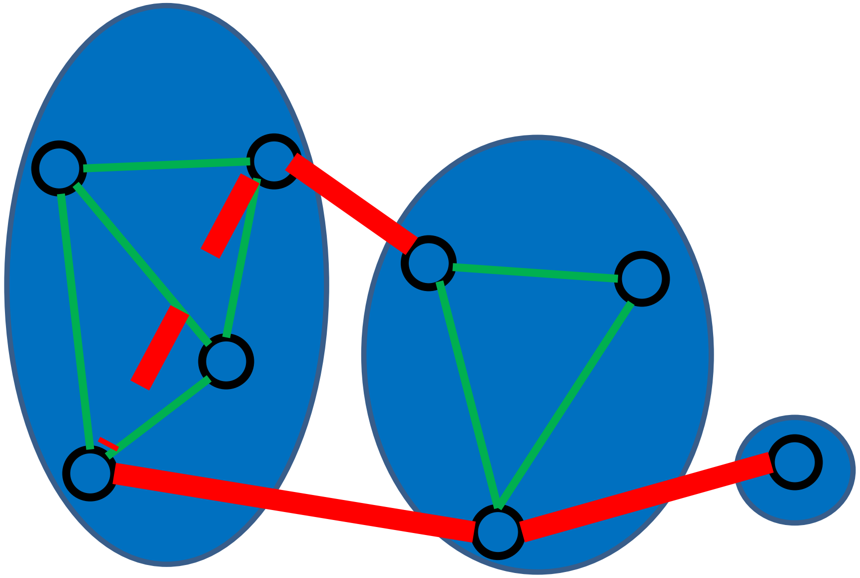
- Inspired by machine learning at WhizBang! LABS
- Practice: [Cohen, McCallum '01, Cohen, Richman '02]
- Theory: [**Blum, Bansal, Chawla '04**]



Correlation Clustering: Example

- **Minimize** # of **incorrectly** classified pairs:

  # Covered non-edges + # Non-covered edges



4 incorrectly classified =
1 covered non-edge +
3 non-covered edges

Approximating Correlation Clustering

- **Minimize** # of **incorrectly** classified pairs
  - ≈ 20000-approximation [Blum, Bansal, Chawla'04]
  - [Demaine, Emmanuel, Fiat, Immorlica'04],[Charikar, Guruswami, Wirth'05], [Ailon, Charikar, Newman'05] [Williamson, van Zuylen'07], [Ailon, Liberty'08],…
  - ≈ 2-approximation [Chawla, Makarychev, Schramm, **Y.** '15]
- **Maximize** # of **correctly** classified pairs
  - $(1-\epsilon)$-approximation [Blum, Bansal, Chawla'04]

Correlation Clustering

One of the most successful clustering methods:

- Only uses **qualitative information** about similarities
- **# of clusters unspecified** (selected to best fit data)
- Applications: document/image **deduplication** (data from crowds or black-box machine learning)
- **NP-hard** [Bansal, Blum, Chawla '04], admits **simple approximation algorithms** with good provable guarantees

Correlation Clustering

More:

- **Survey** [Wirth]
- **KDD'14** tutorial: "Correlation Clustering: From Theory to Practice" [Bonchi, Garcia-Soriano, Liberty] http://francescobonchi.com/CCtuto_kdd14.pdf
- **Wikipedia** article: http://en.wikipedia.org/wiki/Correlation_clustering

Data-Based Randomized Pivoting

3-approximation (expected) [Ailon, Charikar, Newman]

Algorithm:

- Pick a random pivot vertex $\boldsymbol{v}$
- Make a cluster $\boldsymbol{v} \cup N(\boldsymbol{v})$, where $N(\boldsymbol{v})$ is the set of neighbors of $\boldsymbol{v}$
- Remove the cluster from the graph and repeat

Data-Based Randomized Pivoting

- Pick a random pivot vertex $\boldsymbol{p}$
- Make a cluster $\boldsymbol{p} \cup N(\boldsymbol{p})$, where $N(\boldsymbol{p})$ is the set of neighbors of $\boldsymbol{p}$
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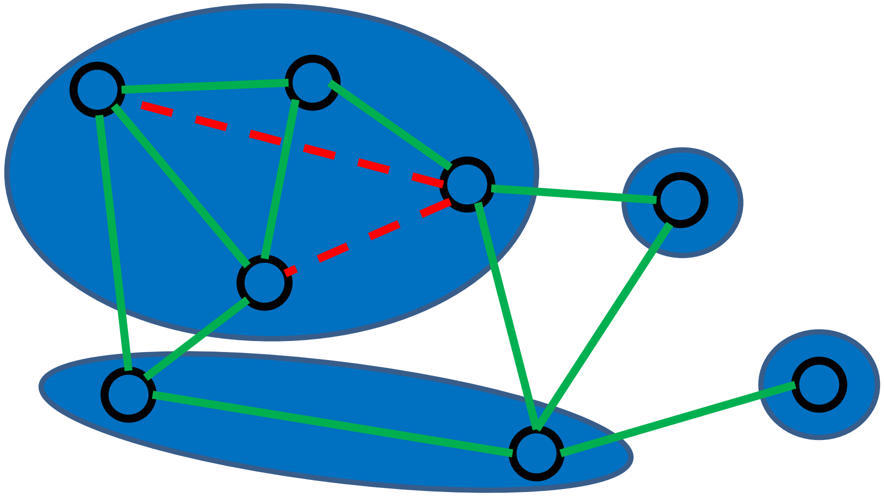


8 incorrectly classified =
2 covered non-edges +
6 non-covered edges

Parallel Pivot Algorithm

- $(3 + \boldsymbol{\epsilon})$-approx. in $O(\log^2 n \,/\, \epsilon)$ rounds [Chierichetti, Dalvi, Kumar, KDD'14]
- Algorithm: while the graph is not empty
  - $\boldsymbol{D}$ = current maximum degree
  - Activate each node independently with prob. $\boldsymbol{\epsilon / D}$
  - Deactivate nodes connected to other active nodes
  - The remaining nodes are **pivots**
  - Create cluster around each pivot as before
  - Remove the clusters

Parallel Pivot Algorithm: Analysis

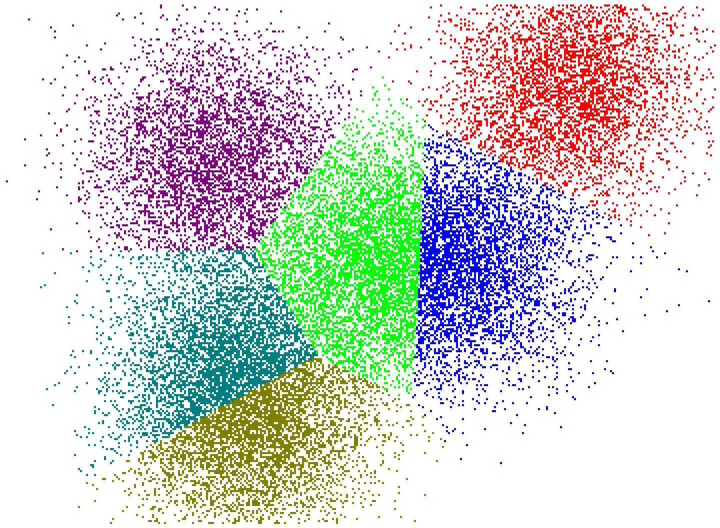
- **Fact:** Halves max degree after $\frac{1}{\epsilon}\log n$ rounds

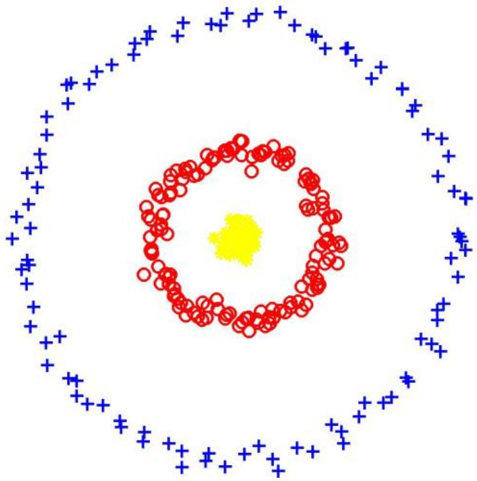
  $\Rightarrow$ terminates in $O\left(\frac{\log^2 n}{\epsilon}\right)$ rounds
- **Fact:** Activation process induces **close to uniform** marginal distribution of the pivots

  $\Rightarrow$ analysis similar to regular pivot gives $(3+\epsilon)$-approximation

Part 2: Clustering Vectors

- Input: $v_1, \ldots, v_{\boldsymbol{n}} \in \mathbb{R}^{\boldsymbol{d}}$
  – Feature vectors in ML, word embedings in NLP, etc.
  – (Implicit) weighted graph of pairwise distances
- Applications:
  – Same as before + Data visualization



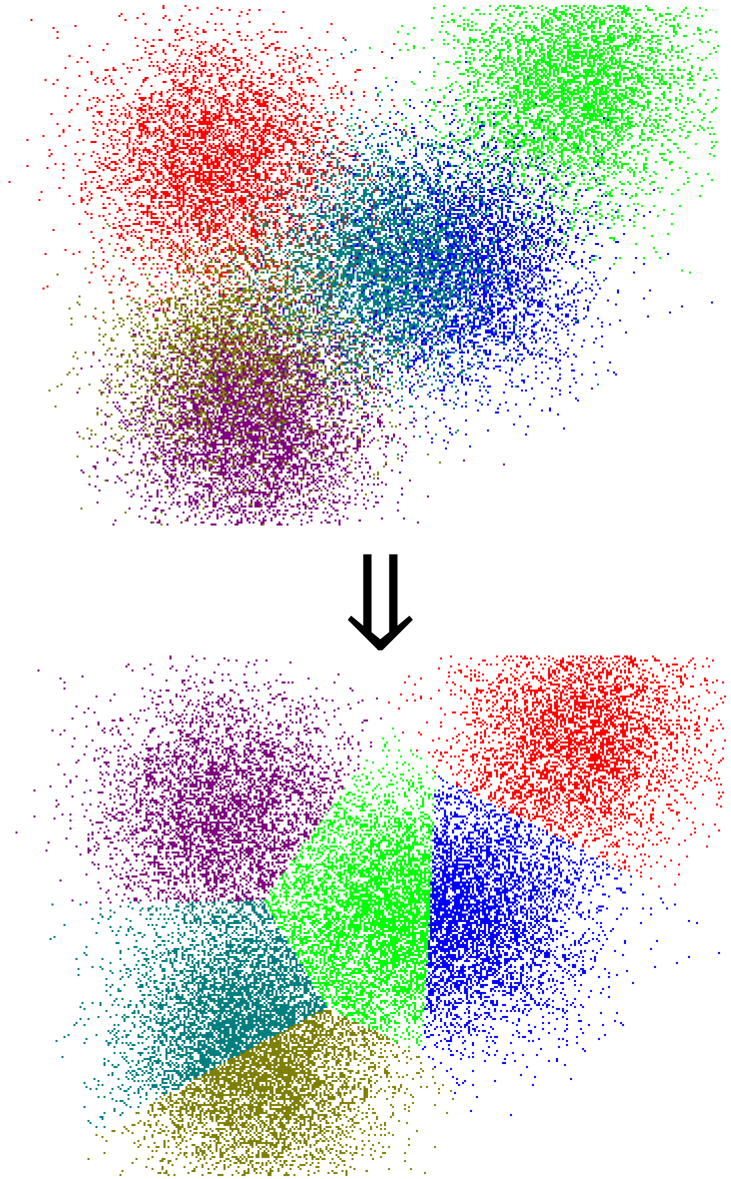


Problem 3: K-means

- Input: $v_1, \dots, v_{\boldsymbol{n}} \in \mathbb{R}^{\boldsymbol{d}}$
- Find $\boldsymbol{k}$ centers $c_1, \dots, c_{\boldsymbol{k}}$
- Minimize sum of squared distance to the closest center:

$$\sum_{i=1}^{n} \min_{j=1}^{k} ||v_i - c_j||_2^2$$

- $||v_i - c_j||_2^2 = \sum_{t=1}^{\boldsymbol{d}} \left(v_{it} - c_{jt}\right)^2$
- NP-hard



K-means++ [Arthur,Vassilvitskii'07]

- $C = \{c_1, \dots, c_t\}$ (collection of centers)
- $d^2(v, C) = \min_{j=1}^{k} ||v - c_j||_2^2$

K-means++ algorithm (gives $O(\log \boldsymbol{k})$-approximation):

- Pick c_1 uniformly at random from the data
- Pick centers $c_2 \dots, c_{\boldsymbol{k}}$ sequentially from the distribution where point v has probability

$$\frac{d^2(v, C)}{\sum_{i=1}^{n} d^2(v_i, C)}$$

K-means|| [Bahmani et al. '12]

- Pick $C = c_1$ uniformly at random from data
- Initial cost: $\psi = \sum_{i=1}^{n} d^2(v_i, c_1)$
- Do $O(\log \psi)$ times:
  - Add $O(\boldsymbol{k})$ centers from the distribution where point v has probability

$$\frac{d^2(v, C)}{\sum_{i=1}^{n} d^2(v_i, C)}$$

- Solve k-means for these O($\boldsymbol{k} \log \psi$) points locally

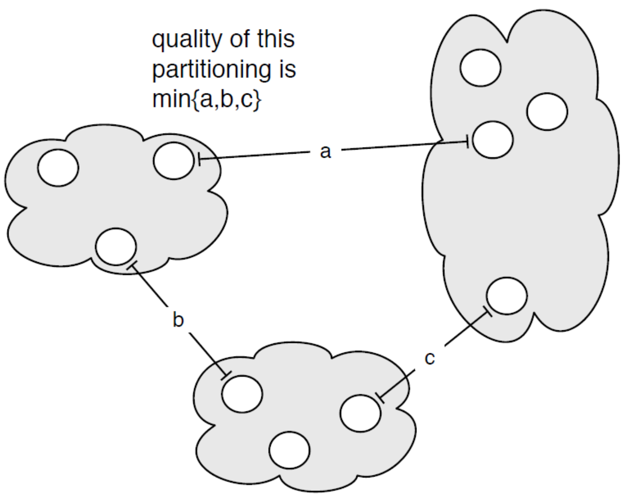
- **Thm.** If final step gives $\boldsymbol{\alpha}$-approximation
  $\Rightarrow O(\boldsymbol{\alpha})$-approximation overall

Problem 4: Single Linkage Clustering

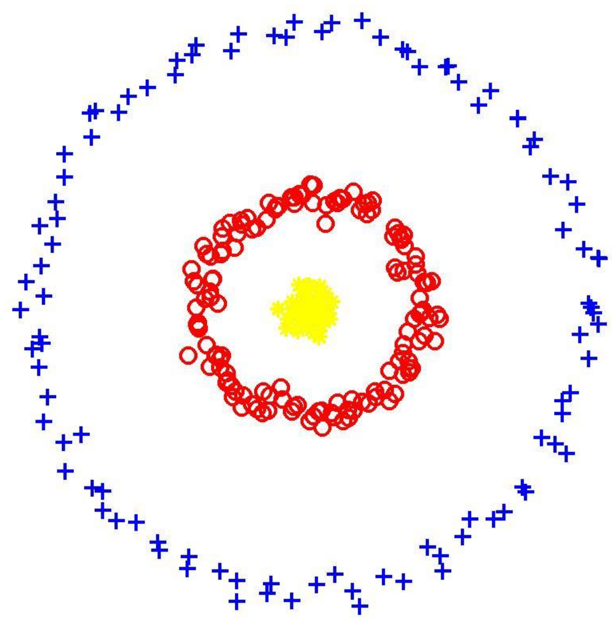
- [Zahn'71] **Clustering** via Minimum Spanning Tree:

k clusters: remove $k-1$ longest edges from MST

- Maximizes **minimum** intercluster distance



[Kleinberg, Tardos]



Large geometric graphs

- Graph algorithms: **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg |V|$.
  - Sparse: $\boldsymbol{S} \ll |V|$.

- **Our setting:**
  - Dense graphs, sparsely represented: O($\mathbf{n}$) space
  - Output doesn't fit on one machine ($\boldsymbol{S} \ll \boldsymbol{n}$)
- **Today:** $(1+\epsilon)$-approximate MST [Andoni, Onak, Nikolov, **Y.**]
  - $\boldsymbol{d} = 2$ (easy to generalize)
  - $\boldsymbol{R} = \log_{\boldsymbol{S}} \boldsymbol{n} = O(1)$ rounds ($\boldsymbol{S} = \boldsymbol{n}^{\boldsymbol{\Omega(1)}}$)

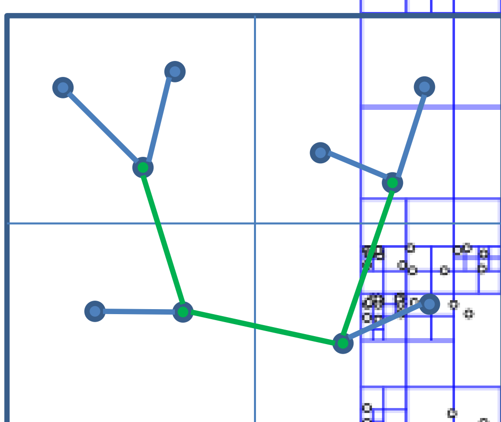
$O(\log n)$-MST in $R = O(\log n)$ rounds

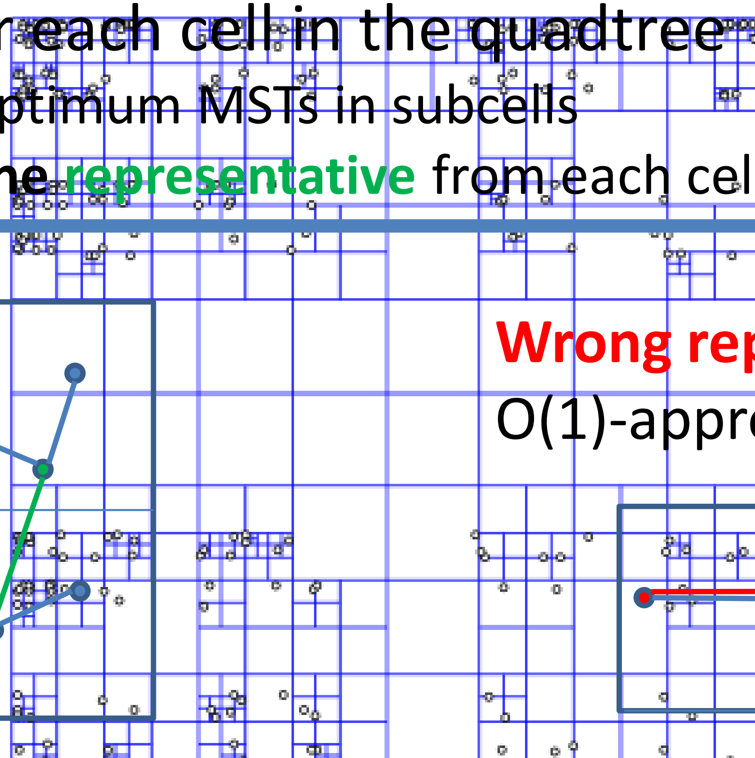
- Assume points have integer coordinates $[0, \ldots, \Delta]$, where $\Delta = O(n^2)$.

Impose an $O(\log n)$-depth quadtree

Bottom-up: For each cell in the quadtree

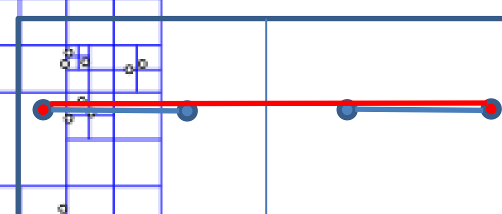
- compute optimum MSTs in subcells
- Use only **one representative** from each cell on the next level





Wrong representative:

O(1)-approximation per level



ϵL-nets

- ϵL-net for a cell C with side length L:

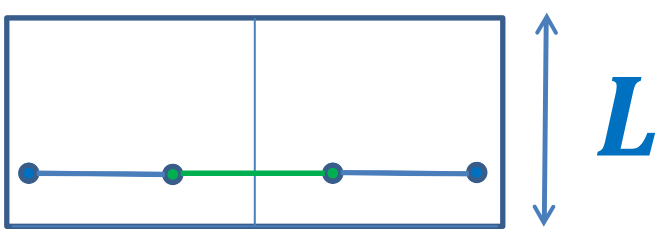
  Collection **S** of vertices in C, every vertex is at distance <= ϵL from some vertex in **S**. (Fact: Can efficiently compute ϵ-net of size $O\left(\frac{1}{\epsilon^2}\right)$)

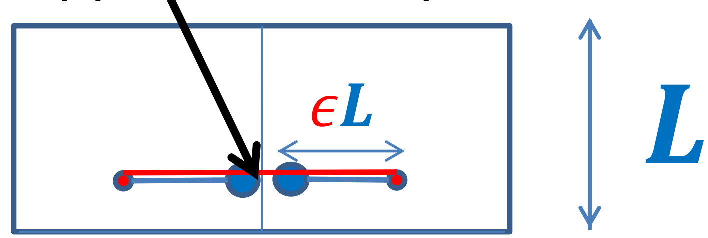
Bottom-up: For each cell in the quadtree
- Compute optimum MSTs in subcells
- Use ϵL-net from each cell on the next level

- **Idea**: Pay only $O(\epsilon L)$ for an **edge** cut by cell with side L
- Randomly shift the quadtree:
  $\Pr[cut\ edge\ of\ length\ \ell\ by\ L] \sim \ell/L$ – charge errors

Wrong representative

O(1)-approximation per level





Randomly shifted quadtree

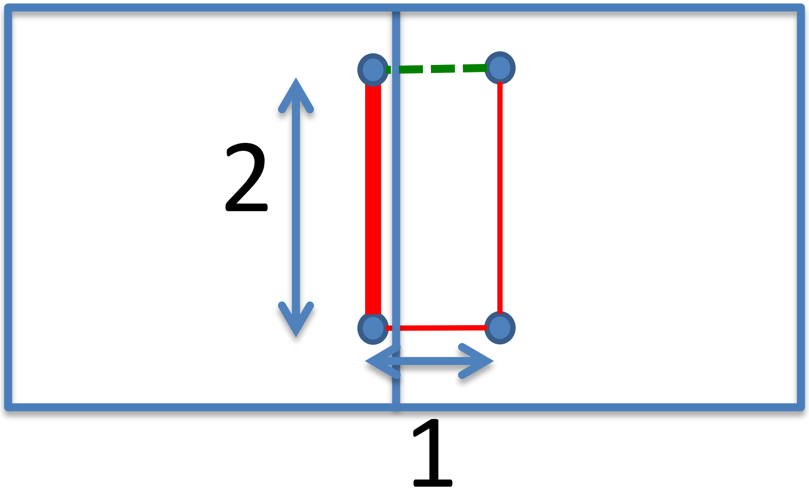
- Top cell shifted by a random vector in $[0, L]^2$

Impose a **randomly shifted** quadtree (top cell length $\mathbf{2\Delta}$)

Bottom-up: For each cell in the quadtree

– Compute optimum MSTs in subcells

– Use ϵL-net from each cell on the next level



Pay **5** instead of **4**

Bad Cut

Pr[Bad Cut] = Ω(1)

$(1+\epsilon)$-MST in $\mathbf{R} = O(\log\ n)$ rounds

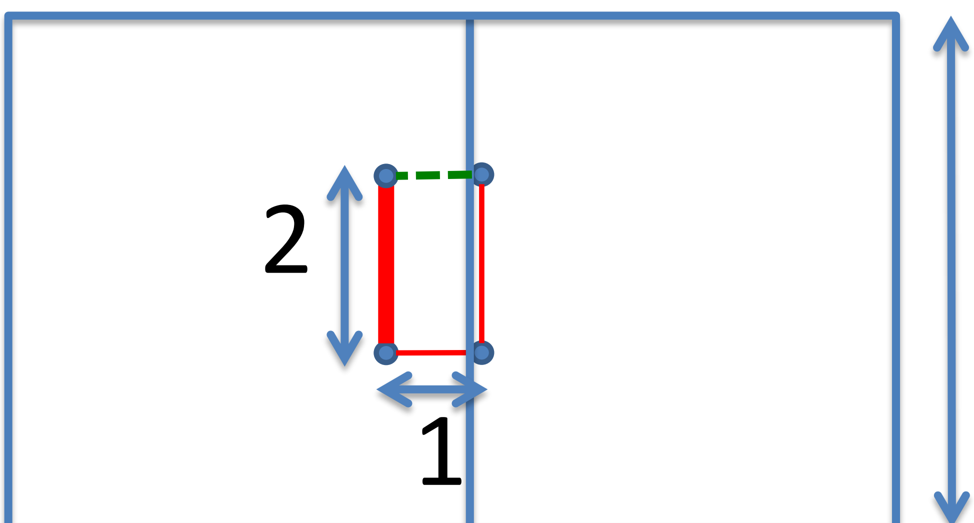
- **Idea:** Only use short edges inside the cells

Impose a **randomly shifted** quadtree (top cell length $\frac{2\Delta}{\epsilon}$)

Bottom-up: For each node (cell) in the quadtree

– compute optimum Minimum Spanning **Forests** in subcells, **using edges of length** $\leq \epsilon L$

– Use only $\epsilon^2 L$-net from each cell on the next level

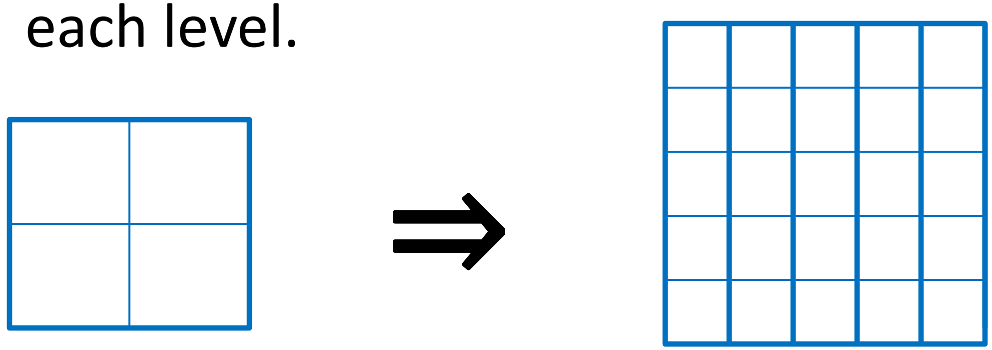


$$L = \Omega(\frac{1}{\epsilon})$$

Pr[Bad Cut] = $O(\epsilon)$

$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

- $O(\log n)$ rounds => $O(\log_s n)$ = O(1) rounds
  - Flatten the tree: $(\sqrt{M} \times \sqrt{M})$-grids instead of (2x2) grids at each level.



$\sqrt{M} = n^{\Omega(1)}$

Impose a **randomly shifted** $(\sqrt{M} \times \sqrt{M})$-tree

Bottom-up: For each node (cell) in the tree

- compute optimum MSTs in subcells via edges of length $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

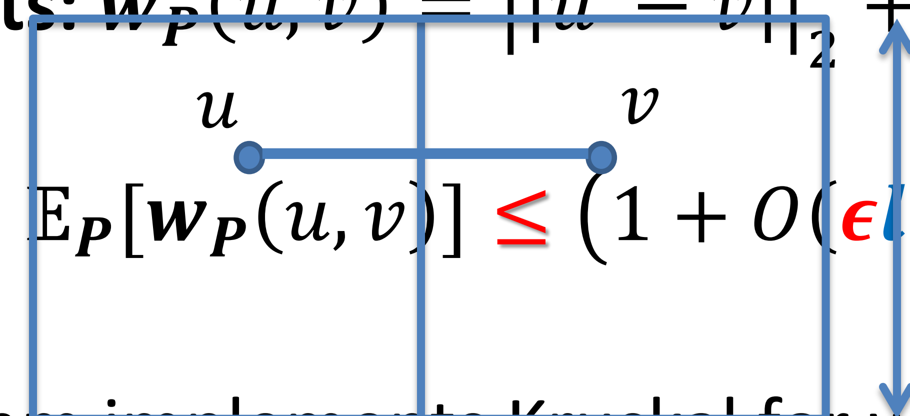
$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

Theorem: Let $\boldsymbol{l}$ = # levels in a random tree $\boldsymbol{P}$

$$\mathbb{E}_{\boldsymbol{P}}[\mathbf{ALG}] \leq \left(1 + O(\boldsymbol{\epsilon l d})\right)\mathbf{OPT}$$

Proof (sketch):

- $\boldsymbol{\Delta_P}(u,v)$ = cell length, which first partitions (u,v)
- **New weights:** $\boldsymbol{w_P}(u,v) = \|u - v\|_2 + \boldsymbol{\epsilon \Delta_P}(u,v)$



$$\|u - v\|_2 \leq \mathbb{E}_{\boldsymbol{P}}[\boldsymbol{w_P}(u,v)] \leq \left(1 + O(\boldsymbol{\epsilon l d})\right)\|u - v\|_2$$

$\boldsymbol{\Delta_P}(u,v)$

- Our algorithm implements Kruskal for weights $\boldsymbol{w_P}$

Technical Details

$(1+\epsilon)$-**MST**:

- "**Load balancing**": partition the tree into parts of the same size
- **Almost linear time locally**: Approximate Nearest Neighbor data structure [Indyk'99]
- Dependence on dimension **d** (size of ϵ-net is $O\left(\frac{d}{\epsilon}\right)^{d}$)
- Generalizes to bounded **doubling dimension**